

Maximal switchability of centralized networks

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Abstract

We consider continuous time Hopfield-like recurrent networks as dynamical models for gene regulation and neural networks. We are interested in networks that contain n high-degree nodes preferably connected to a large number of N_s weakly connected satellites, a property that we call n/N_s -centrality. If the hub dynamics is slow, we obtain that the large time network dynamics is completely defined by the hub dynamics. Moreover, such networks are maximally flexible and switchable, in the sense that they can switch from a globally attractive rest state to any structurally stable dynamics when the response time of a special controller hub is changed. In particular, we show that a decrease of the controller hub response time can lead to a sharp variation in the network attractor structure: we can obtain a set of new local attractors, whose number can increase exponentially with N , the total number of nodes of the network. These new attractors can be periodic or even chaotic. We provide an algorithm, which allows us to design networks with the desired switching properties, or to learn them from time series, by adjusting the interactions between hubs and satellites. Such switchable networks could be used as models for context dependent adaptation in functional genetics or as models for cognitive functions in neuroscience.

Keywords Networks, Attractors, Chaos, Bifurcations

1 Introduction

Networks of dynamically coupled elements have imposed themselves as models of complex systems in physics, chemistry, biology and engineering [36]. The most studied propriety of networks is their topological structure. Structural features of networks are usually defined by the distribution of the number of direct connections a node has, or by various statistical properties of paths and circuits in the network [36, 2]. An important structure related property of networks is their scale-freeness [23, 22, 2, 6] often invoked as a paradigm of self-organization and spontaneous emergence of complex collective behaviour [9]. In scale-free networks the fraction $P(k)$ of nodes in the network having k connections to other nodes (i.e. having degree k) can be estimated for large values of k as $P(k) \sim k^{-\gamma}$, where γ is a parameter whose value is typically in the range $2 < \gamma < 3$ [2]. In such networks, the degree is extremely heterogeneous. In particular, there are strongly connected nodes that can be named hubs, or centers. The hubs communicate to each other directly, or via a number of weakly connected nodes. The weakly connected nodes that interact mainly with hubs can be called satellites. Scale-free networks have also nodes of intermediate connectivity. Networks that have only two types of nodes, strongly connected hubs and weakly connected satellites are known as bimodal degree networks [51]. Because of the presence of a large number of hubs, scale-free

or bimodal degree networks can be called centralized. Centralized connectivity has been found by functional imaging of brain activity in neuroscience [9], and also by large scale studies of the protein-protein interactions or of the metabolic networks in functional genetics [23, 22].

The centralized architecture was shown to be important for many emergent properties of networks. For instance, there has been a lot of interest in the resilience of networks with respect to attacks that remove some of their components [3]. It was shown that networks with bimodal degree connectivity are resilient to simultaneous targeted and random attacks [51], whereas scale-free networks are robust with respect to random attacks, but sensitive to targeted attacks that are directed against hubs [10, 4]. For this reason, the term "robust-yet-fragile" was coined in relation to scale-free networks [7].

From a more dynamical perspective, a centralized architecture facilitates communication between hubs, stabilizes hubs by making them insensitive to noise [55, 54] and allows for hub synchronization even in the absence of satellite synchronization [42, 41, 48]. Another important question concerning networks is how to push their dynamics from one region of the phase space to another or from one type of behaviour to another, briefly how to control the network dynamics [30, 49, 35, 12, 46, 40, 24, 39, 16, 61]. Several authors used Kalman's results for linear systems to understand how network structure influences network dynamics controllability, and in particular how to choose the control nodes [30, 35, 12]. As pointed out by [34, 27] several difficulties occur when one tries to apply these general results to real networks. Even for linear networks, the control of trajectories is nonlocal [49] and shortcuts are rarely allowed. As a result, even small changes of the network state may ask for control signals of large amplitude and energy [59]. The control of nonlinear networks is even more difficult and in this case we have no general results. Nonlinear networks can have several co-existing attractors and it is interesting to find out how to push the state of the network from one attractor basin to another. The ability of networks to change attractor under the effect of targeted perturbations can be called switchability. In relation to this, the paper [43] has introduced the terminology "stable yet switchable" (SyS) meaning that the network remains stable given a context and is able to reach another stable state when a stimulus indicates a change of the context. It was shown, by numerical simulations, that centralized networks with bimodal degree distribution are more prone to SyS behavior than scale-free networks [43]. Switchability is important for practical reasons, for instance in drug design. In such applications, one uses pharmaceutical action on nodes to push a network that functions in a pathological attractor (such pathological attractors were discussed in relation to cancer [21] or neurological disorders [47, 14]) to a healthy functioning mode, characterized by a different attractor. Numerical methods to study switchability of linear [58] and nonlinear [11] networks were discussed in relation with drug design in cancer research. In theoretical biology, network switchability can be important for mathematical theories of genetic adaptation [37]. If one looks at organisms as complex systems and model them by networks, then adaptation to changes in the environment can be described as switching the network from one attractor to another one with a higher fitness [37]. An important question that is often asked with respect to tuning network dynamics is how many driver nodes are needed to control that dynamics. For linear networks, it was shown that this number is large if we aim to obtain a total control, which allows us to switch the network between any pair of states. This number can be as high as 80% for molecular regulatory networks [31]. This fact, as emphasized in [58], contradicts empirical results about cellular reprogramming and about adaptive evolution. Much less nodes are needed if instead of full controllability one wants switching between specific pairs of unexpected and desired states [58]. This concept, named "transittability" in [58], is very similar to our switchability, but was studied only for linear systems.

In this paper, we study dynamical properties of large nonlinear networks with centralized architecture. We consider continuous time versions of the Hopfield model of recurrent neural networks [19] with a large number N of neurons. The Hopfield model is based on the two-states McCulloch and Pitts formal neuron and uses symmetrical weight matrices to specify interactions between neurons. Like to the Hopfield version, we use a thresholding function to describe switching between

the two neuron states, active and inactive. However, contrary to the original Hopfield version, we do not impose symmetrical interactions between neurons, in other words our weight matrix is not necessarily symmetric. This model has been successfully used to describe associative memories [19], neural computation [20, 32], disordered systems in statistical physics [50], neural activity [29, 14] and also to investigate space-time dynamics of gene networks in molecular biology [33, 57]. The choice of such type of dynamics is motivated by the existence of universal approximation results for multilayered perceptrons (see, for example, [5]). In particular, we have shown elsewhere that networks with Hopfield-type dynamics can approximate any structurally stable dynamics, including reaction-diffusion biochemical networks also largely used in biology [54].

Our aim is to study analytically the ability of a network with centralized architecture to be switchable. We employ a special notion of centrality. Many biological networks exhibit so-called dissortative mixing, i.e., high-degree nodes are preferably connected to low-degree nodes [25]. We will consider networks with n strongly connected hubs. We also assume that each hub is under the action of at least N_s weakly connected satellites, that on turn receive actions from all the hubs. For large networks, N_s increases at least as fast as a power of N , $N_s > c_0 N^\theta$ where $c_0 > 0$, $0 < \theta < 1$ are constants and N is the total number of nodes. We call this property n/N_s -centrality. This network architecture ensures a large number of feed-back loops that produce complex dynamics. Furthermore, the dissortative connectivity implies functional heterogeneity of the hubs and satellites. The hubs play the role of controllers and the satellites sustain the feedback loops needed for attractor multiplicity. The large number of satellites guarantees a sufficient flexibility of the network dynamics and also buffer the perturbations transmitted to the hubs. This principle applies well to gene networks. The hubs in such networks can be the transcription factors, which are stabilized by numerous interactions with non-coding RNAs that represent the satellites [28]. In addition to structural conditions, we will consider a special correlation between time scales and connectivity of the nodes: the hubs have slow response, whereas the satellites respond rapidly. This condition is natural for many real networks. The hubs have to cope with multiple tasks, therefore they must have more complex interaction than the satellites. Consequently, the hubs need more resources to be produced, decomposed, and react with other nodes, therefore their dynamics is slow. This property is obvious for gene networks, where transcription factors are complex proteins, much larger and more stable than the non-coding RNAs.

Our first result is valid without conditions on the structure and depends only on the condition on the timescales. We assume that there exist $n \ll N$ slow nodes, whereas all the remaining ones are fast. Then, the dynamics of the network can be reduced to n variables. We prove the existence of an inertial manifold of dimension n , which completely captures all network dynamics for large times. We recall that the fundamental concept of inertial manifold was introduced for infinite dimensional and multidimensional systems. The inertial manifolds are globally attracting invariant ones [38]. The large time dynamics of a system possessing an inertial manifold, is defined by a smooth vector field F of relatively small dimension, so-called inertial form. All attractors lie on inertial manifold [38].

The second result holds under the structural assumption that the network is n/N_s -central. Under this condition, we show that the inertial forms F obtained from such networks are dense in the set of all smooth vector fields of dimension n . This implies that given a certain combination of attractors defined by vector fields Q_i we can construct a centralized network that exhibits a combination of attractors that is topologically equivalent to the one given. Furthermore, we show that n/N_s -central networks can exhibit "maximal switchability". By changing a control parameter ξ , which determines the response time of a single network hub ("controller" hub), we can sharply change the network attractor. For instance we can switch from a situation when the network has a single rest point for $\xi > \xi_0$ to a situation when the network has a complicated global attractor for $\xi < \xi_0$, including a number of local attractors, which may be periodic or chaotic. The network state tends to the corresponding local attractor depending on the initial state of the control hub. This result shows in an analytical and rigorous way how nonlinear networks can be switched by only

one control node. The possibility of switching nonlinear networks by a small number of nodes is crucial in theories of genetic adaptation. Indeed, phenomenological theories predict and empirical data confirm that the main part of the adaptive evolution process consists in only a few mutations producing large fitness changes [37].

Our third result proves, in an analytical way, that the number of rest point local attractors (and therefore the network capacity) of n/N_s -central networks may be exponentially large in the number of nodes.

We also describe a constructive algorithm, which allows us to obtain a centralized network that performs a prescribed inertial dynamics and the desired switching properties of the network.

2 Problem statement and main assumptions

We consider the Hopfield-like networks [19] described by the ordinary differential equations

$$\frac{du_i}{dt} = \sigma\left(\sum_{j=1}^N W_{ij}u_j - h_i\right) - \lambda_i u_i, \quad (2.1)$$

where u_i , h_i and $\lambda_i > 0$, $i = 1, \dots, N$ are node activities, activation thresholds and degradation coefficients, respectively. The matrix entry W_{ij} describes the action of the node j on the node i , which is an activation if $W_{ij} > 0$ or a repression if $W_{ij} < 0$. Contrary to the original Hopfield model, the interaction matrix W is not necessarily symmetric. The function σ is an increasing and smooth (at least twice differentiable) "sigmoidal" function such that

$$\sigma(-\infty) = 0, \quad \sigma(+\infty) = 1, \quad \sigma'(z) > 0. \quad (2.2)$$

Typical examples can be given by

$$\sigma(h) = \frac{1}{1 + \exp(-h)}, \quad \sigma(h) = \frac{1}{2} \left(\frac{h}{\sqrt{1+h^2}} + 1 \right). \quad (2.3)$$

The structure of interactions in the model is defined by a weighted digraph (V, E, W) with the set V of nodes, the edge set E and weights W_{ij} . The nodes v_j , $j = 1, \dots, N$ can be neurons or genes, depending on applications.

Assumption 1.

Assume that if $W_{ji} \neq 0$, then (i, j) is an edge of the graph, $(i, j) \in E$. This means that the i -th node can act on the j -th node only if it is prescribed by an edge of the digraph (V, E, W) . We also suppose that $(i, i) \notin E$, i.e., the nodes do not act on themselves.

Assume that the digraph (V, E, W) satisfies a condition, which is a variant of the centrality property. This condition is a purely topological one and thus it is independent on the weights W_{ij} . To formulate this condition, we introduce a special notation.

Let us consider a node v_j . Let us denote by $S^*(j)$ the set of all nodes, which act on the neuron j :

$$S^*(j) = \{v_i \in V : \text{edge } (i, j) \in E\}. \quad (2.4)$$

For each set of nodes $\mathcal{C} \subset V$ we introduce the set $\mathcal{S}(\mathcal{C})$ of the nodes, which are under action of all nodes from \mathcal{C} and which are not belonging to \mathcal{C} :

$$\mathcal{S}(\mathcal{C}) = \{v_i \in V : \text{for each } j \in \mathcal{C} \text{ edge } (j, i) \in E \text{ and } v_i \notin \mathcal{C}\}. \quad (2.5)$$

n/N_s -Centrality assumption. *The graph (V, E, W) is connected and there exists a set of nodes \mathcal{C} such that*

- i \mathcal{C} consists of n nodes;
- ii for each $j \in \mathcal{C}$ the intersection $\mathcal{S}^*(j) \cap \mathcal{S}(\mathcal{C})$ contains at least N_s nodes, where $N_s > c_0 N^\theta$ with constants $c_0 > 0, \theta \in (0, 1)$, which are independent of j and N .

The nodes from \mathcal{C} can be interpreted as hubs (centers) and the nodes from $\mathcal{S}(\mathcal{C})$ are the satellites. The condition ii implies that each center is under action of sufficiently many satellites. In turn, if we consider the union of these satellites, all the centers act on them (see Fig.1). Such an intensive interaction leads, as we will see below, to a very complicated large time behaviour.

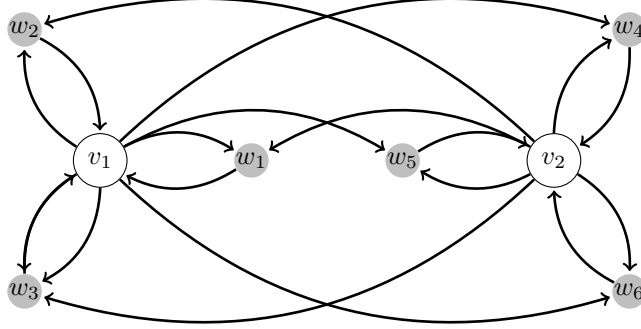


Figure 1: This image shows an n/N_s -central network with $n = 2$ and $N_s = 3$. The graph consists of 8 nodes denoted by $v_1, v_2, w_1, w_2, w_3, w_4, w_5, w_6$. The set $\{v_1, v_2\}$ is the set of centers \mathcal{C} . The sets $\mathcal{S}(\mathcal{C}), \mathcal{S}^*(v_1)$ and $\mathcal{S}^*(v_2)$ are as follows: $\mathcal{S}(\mathcal{C}) = \{w_1, w_2, w_3, w_4, w_5, w_6\}$, $\mathcal{S}^*(v_1) = \{w_1, w_2, w_3\}$ and $\mathcal{S}^*(v_2) = \{w_4, w_5, w_6\}$. The sets $\mathcal{S}^*(v_1) \cap \mathcal{S}(\mathcal{C}) = \{w_1, w_2, w_3\}$ and $\mathcal{S}^*(v_2) \cap \mathcal{S}(\mathcal{C}) = \{w_4, w_5, w_6\}$ contain three nodes each.

3 Outline of main results

Our results can be outlined as follows. The result on the inertial dynamics existence describes a situation, when the interaction topology is quite arbitrary. We assume that there exist n slow nodes, say, u_1, u_2, \dots, u_n with $\lambda_i = O(1)$ whereas all the rest ones u_{n+1}, \dots, u_N are fast, i.e., the corresponding λ_i have order $O(\kappa^{-1})$, where κ is a small parameter. Then we show that there exists an inertial manifold of dimension n . We obtain, under general conditions, that for times $t \gg \kappa \log \kappa$ the dynamics of (2.1) is defined by the reduced equations

$$\frac{du_j}{dt} = F_j(u_1, \dots, u_n, W, h, \lambda), \quad (3.1)$$

$$u_k = U_k(u_1, \dots, u_n, W, h, \lambda), \quad k = n + 1, \dots, N, \quad (3.2)$$

where F_j and U_k are some smooth functions of u_1, \dots, u_n , and h, λ denote the vector parameters (h_1, \dots, h_N) and $(\lambda_1, \dots, \lambda_N)$, respectively. So, F gives us the inertial form on an inertial manifold. The inertial form completely defines the dynamics for large times [38].

More interestingly, we can show that the vector field F is, in a sense, maximally flexible. Roughly speaking, by the number of nodes N , the matrix W and h we can obtain all possible fields F (up to a small accuracy ϵ , which can be done arbitrarily small as N goes to ∞), see section 5 for a formal statement of this flexibility property. For the networks this flexibility property holds under n/N_s -Centrality assumption.

Let us introduce a special control parameter ξ , which modulates the degradation coefficient λ_i for a hub: $\lambda_i = \xi \bar{\lambda}_i$ for some $i \in \mathcal{C}$. This hub is a "controller". When we vary the coefficient ξ , the interaction topology and the entries of the interaction matrix do not change, but the response time of the controller hub changes.

One can choose the network parameters N, W, λ in such a way that for $\xi > \xi_0$ the global attractor is trivial, it is a rest point, but for an open set of other values ξ the global attractor of (2.1) contains a number of local attractors.

This result can be interpreted as "maximal switchability". A similar effect was found in [13] by numerical simulations for some models of neural networks. This effect describes a transition from neural resting states (NRS) to complicated global attractors, which occur as a reaction on learning tasks. Note that in [13] attractors consist of a number of steady states. In our case the global attractors can include many local attractors of all possible kinds including chaotic and periodic ones.

We end this section with a remark. Our method approximates vector fields by neural networks, but what can be said about the relationship between the trajectories of the simulated system and the ones corresponding to the neural network?

For chaotic and even for periodic attractors, direct comparison of trajectories is not a suitable test for the accuracy of the approximation. General mathematical arguments allow us say only that these trajectories will be close for bounded times. For large times we can say nothing especially for general chaotic attractors. Consider the case when the attractor \mathcal{A} of the simulated system is transitive. This means the dynamics is ergodic and for smooth function ϕ the time averages

$$S_{F,\phi} = \lim_{T \rightarrow +\infty} T^{-1} \int_0^T \phi(v(t)) dt \quad (3.3)$$

coincide with the averages $\int_{\mathcal{A}} \phi(v) d\mu(v)$ over the attractor, where μ is an invariant measure on \mathcal{A} .

Then, a suitable criterion of approximation is that the averages $S_{F,\phi}$ and the corresponding ones generated by the approximating centralized neural network, are close for smooth ϕ :

$$|S_{F,\phi} - S_{G_{anN},\phi}| = Err_{approx} < \delta(\epsilon, \phi) \quad (3.4)$$

where G_{anN} is the neural network approximation of F and $\delta \rightarrow 0$ as $\epsilon \rightarrow 0$. This "stochastic stability" property holds for hyperbolic (structurally stable) attractors [26, 60, 56].

4 Conditions on network parameters and attractor existence

Our first results do not use any assumptions on the network topology. However, we suppose that there are two types of network components that are distinguished by their time scales into slow nodes and fast nodes. To take into account the two types of the nodes, we use distinct variables v_j for slow variables, $j = 1, \dots, n$ and w_i for the fast ones, $i = 1, \dots, N - n = N_1$. The real matrix entry A_{ji} defines the intensity of the action of the fast node i on the slow node j . Similarly, the $n \times N_1$ matrix \mathbf{B} , $N_1 \times N_1$ matrix \mathbf{C} and $n \times n$ matrix \mathbf{D} define the action of the slow nodes on the fast ones, the interactions between the fast nodes and the interactions between the slow nodes, respectively. We denote by h_i and λ_i the threshold and degradation parameters of the fast nodes and by \tilde{h}_i and $\tilde{\lambda}_i$ the same parameters for the slow nodes, respectively. To simplify formulas, we use the notation

$$\sum_{j=1}^n D_{ij} v_j = \mathbf{D}_i v, \quad \sum_{k=1}^N C_{jk} w_k = \mathbf{C}_j w.$$

Then, equations (2.1) can be rewritten as follows:

$$\frac{dw_i}{dt} = \sigma \left(\mathbf{B}_i v + \mathbf{C}_i w - \tilde{h}_i \right) - \kappa^{-1} \tilde{\lambda}_i w_i, \quad (4.1)$$

$$\frac{dv_j}{dt} = \sigma \left(\mathbf{A}_j w + \mathbf{D}_j v - h_j \right) - \lambda_j v_j, \quad (4.2)$$

where $i = 1, \dots, N_1$, $j = 1, \dots, n$. Here unknown functions $w_i(t), v_j(t)$ are defined for times $t \geq 0$. We assume that κ is a positive parameter, therefore, the variables w_i are fast.

We set the initial conditions

$$w_i(0) = \tilde{\phi}_i \geq 0, \quad v_j(0) = \phi_j \geq 0. \quad (4.3)$$

It is natural to assume that all concentrations are non-negative at the initial moment. It is clear that they stay non-negative for all times.

4.1 Global attractor exists

Let us prove that the network dynamics is correctly defined for all t and solutions are non-negative and bounded. For positive vectors $r = (r_1, \dots, r_n)$ and $R = (R_1, \dots, R_{N_1})$, let us introduce the sets \mathcal{B} defined by

$$\mathcal{B}(r, R) = \{(w, v) : 0 \leq v_j \leq r_j, \ 0 \leq w_i \leq R_i, \ j = 1, \dots, n, \ i = 1, \dots, N_1\}.$$

Note that

$$\frac{dw_i}{dt} < 1 - \kappa^{-1} \tilde{\lambda}_i w_i.$$

Thus, $w_i(t) < X(t)$ for positive times t , where

$$\frac{dX}{dt} = 1 - \kappa^{-1} \tilde{\lambda}_i X, \quad X(0) = w_i(0).$$

Therefore, resolving the last equation, and repeating the same estimates for $v_i(t)$, one finds

$$\begin{aligned} 0 \leq w_i(x, t) &\leq \tilde{\phi}_i \exp(-\tilde{\kappa}^{-1} \tilde{\lambda}_i t) + \kappa \tilde{\lambda}_i^{-1} (1 - \exp(-\kappa^{-1} \tilde{\lambda}_i t)), \\ 0 \leq v_j(x, t) &\leq \phi_j \exp(-\lambda_j t) + \lambda_j^{-1} (1 - \exp(-\lambda_j t)), \end{aligned} \quad (4.4)$$

Let us take arbitrary $a > 1$ and let $r_j(a) = a \lambda_j^{-1}$ and $R_i(a) = a \kappa \tilde{\lambda}_i^{-1}$. Estimates (4.4) show that solutions of (4.1), (4.2) exist for all times t and they enter the set $\mathcal{B}(r(a), R(a))$ at a time moment t_0 . The solutions stay in this set for all $t > t_0$, thus, this set is absorbing. This shows that system (4.1),(4.2) defines a global dissipative semiflow S_H^t [17]. Moreover, this semiflow has a global attractor contained in each $\mathcal{B}(r(a), R(a))$, where $a > 1$.

4.2 Assumptions for slow/fast networks.

A simpler asymptotic description of system dynamics is possible under assumptions on network components timescales. We suppose here that the u -variables are fast and the v -ones are slow. We show then that the fast w variables are slaved, for large times, by the slow v modes. More precisely, one has $w = \kappa U(v) + \tilde{w}$, where $\kappa U(v)$ is a correction and $\kappa > 0$ is a small parameter. This means that, for large times, the fast nodes dynamics is completely controlled by the slow nodes.

To realize this approach, let us assume that the system parameters $\mathbf{P} = \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, h, \tilde{h}, \tilde{\lambda}, \lambda\}$ satisfy the following conditions:

$$\mathbf{A} = \kappa^{-1} \tilde{\mathbf{A}}, \quad (4.5)$$

$$|\bar{\mathbf{A}}|, |\mathbf{B}|, |\mathbf{C}|, |\mathbf{D}| < c_0, \quad (4.6)$$

$$0 < c_1 < \bar{\lambda}_i < c_2, \quad 0 < \tilde{\lambda}_i < c_3. \quad (4.7)$$

Here all positive constants c_k are independent of κ for small κ .

The scaling assumption on \mathbf{A} is needed because, as we will prove later, $w = \mathcal{O}(\kappa)$ for small κ . For the same reasons, $\mathbf{C}_i w$ can be neglected with respect to $\mathbf{B}_i v$ for small κ , meaning that the action of centers on satellites is dominant with respect to satellites mutual interactions. In other words, these conditions describe a divide and rule control principle .

5 Realization of prescribed dynamics and maximally flexible systems

Our goal is to show that the network dynamics can realize, in a sense, arbitrary structurally stable dynamics of the centers. To precise this assertion, let us describe the method of realization of the vector fields for dissipative systems (proposed in [44]). More precisely, we are interested in systems enjoying the following properties:

A *These systems generate global semiflows $S_{\mathcal{P}}^t$ in an ambient Hilbert or Banach phase space H . These semiflows depend on some parameters \mathcal{P} (which could be elements of another Banach space \mathcal{B}). They have global attractors and finite dimensional local attracting invariant C^1 - manifolds \mathcal{M} , at least for some \mathcal{P} .*

B *Dynamics of $S_{\mathcal{P}}^t$ reduced on these invariant manifolds can be, in a sense, almost completely tuned by variations of the parameter \mathcal{P} .*

It can be described as follows. Assume the differential equations

$$\frac{dq}{dt} = Q(q), \quad Q \in C^1(B^n) \quad (5.1)$$

define a global semiflow in a unit ball $B^n \subset \mathbb{R}^n$.

For any prescribed dynamics (5.1) and any $\epsilon > 0$, we can choose suitable parameters $\mathcal{P} = \mathcal{P}(n, F, \epsilon)$ such that

B1 *The semiflow $S_{\mathcal{P}}^t$ has a C^1 - smooth locally attracting invariant manifold $\mathcal{M}_{\mathcal{P}}$ diffeomorphic to B^n ;*

B2 *The reduced dynamics $S_{\mathcal{P}}^t|_{\mathcal{M}_{\mathcal{P}}}$ is defined by equations*

$$\frac{dq}{dt} = \tilde{Q}(q, \mathcal{P}), \quad \tilde{Q} \in C^1(B^n) \quad (5.2)$$

where the estimate

$$|Q - \tilde{Q}|_{C^1(B^n)} < \epsilon \quad (5.3)$$

holds. In other words, one can say that, by \mathcal{P} , the reduced dynamics on the invariant manifold can be specified to within an arbitrarily small error.

Therefore, roughly speaking all robust dynamics (stable under small perturbations) can be generated by the systems, which satisfy above formulated properties. Such systems can be named *maximally flexible*. In order to show that maximal flexibility covers also the case of chaotic dynamics, let us recall some facts about chaos and hyperbolic sets.

Let us consider dynamical systems (global semiflows) S_1^t, \dots, S_k^t , $t > 0$, defined on the n -dimensional closed ball $B^n \subset \mathbb{R}^n$ defined by finite dimensional vector fields $F^{(k)} \in C^1(B^n)$ and having structurally stable attractors \mathcal{A}_l , $l = 1, \dots, k$. These attractors can have a complex form, since it is well known that structurally stable dynamics may be “chaotic”. There is a rather wide variation in different definitions of “chaos”. In principle, one can use here any concept of chaos,

provided that this is stable under small C^1 -perturbations. To fix ideas, we shall use here, following [45], such a definition. We say that a finite dimensional dynamics is chaotic if it generates a compact invariant hyperbolic set Γ , which is not a periodic cycle or a rest point (for a definition of hyperbolic sets see, for example, [45]). The hyperbolic sets give remarkable analytically tractable examples, where chaotic dynamics can be studied. For example, the Smale horseshoe is a hyperbolic set. If this set Γ is attracting we say that Γ is a chaotic (strange) attractor. In this paper, we use only the following basic property of hyperbolic sets, so-called Persistence [45]. This means that the hyperbolic sets are, in a sense, stable(robust). This property can be described as follows. Let a system of differential equations be defined by a C^1 -smooth vector field Q on an open domain in \mathbb{R}^n with a smooth boundary or on a smooth compact finite dimensional manifold. Assume this system defines a dynamics having a compact invariant hyperbolic set Γ . Let us consider ϵ -perturbed the vector field $Q + \epsilon\tilde{Q}$, where \tilde{Q} is bounded in C^1 -norm. Then, if $\epsilon > 0$ is sufficiently small, the perturbed field also generates dynamics with another compact invariant hyperbolic set $\tilde{\Gamma}$. The corresponding dynamics restricted to Γ and $\tilde{\Gamma}$ respectively, are topologically orbitally equivalent (topological equivalency of two semiflows means that there exists a homeomorphism, which maps the trajectories of the first semiflows on the trajectories of the second one, see [45] for details).

We recall that chaotic structurally stable (persistent) attractors and invariant sets exist: this fact is well known from the theory of hyperbolic dynamics [45].

Thus, any kind of the chaotic hyperbolic sets can occur in the dynamics of the systems, for example, the Smale horseshoes, Anosov flows, and the Ruelle-Takens-Newhouse chaos, see [45]. Examples of systems satisfying these properties can be presented by some reaction-diffusion equations and systems [44, 52, 53], and neural network models [53].

6 Main results

For vectors $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ such that $a_i < b_i$ for each i let us denote by

$$\Pi(a, b) = \{v \in \mathbb{R}^n : a_i \leq v_i \leq b_i\} \quad (6.1)$$

a n -dimensional box in v -space. Moreover, let us define Π_λ by $\Pi_\lambda = \Pi(0, \lambda^{-1})$, where the vector λ^{-1} has components $(\lambda_1^{-1}, \dots, \lambda_n^{-1})$.

Theorem 6.1 *Under assumptions (2.2), (4.5), (4.6) and (4.7) for sufficiently small κ there exists a n -dimensional inertial manifold \mathcal{M}_n defined by*

$$w_i = \kappa \tilde{\lambda}_i^{-1} U_i(v, \kappa, \mathbf{P}), \quad v \in \Pi_\lambda \quad (6.2)$$

where $U_i \in C^{1+r}(\Pi_\lambda)$, and $r \in (0, 1)$. The functions U_i admit the estimate

$$|U_i(v, \kappa, \mathbf{P}) - \sigma(\mathbf{B}_i v - \tilde{h}_i)|_{C^1(\Pi_\lambda)} < c_4 \kappa, \quad v \in \Pi_\lambda. \quad (6.3)$$

The v dynamics for large times takes the form

$$\frac{dv_j}{dt} = F_j(v, \mathbf{P}) + \tilde{F}_j(v, \kappa, \mathbf{P}), \quad (6.4)$$

where \tilde{F}_j satisfy

$$|\tilde{F}_j|_{C^1(\Pi_\lambda)} < c_6 \kappa \quad (6.5)$$

with

$$F_j(v, \mathbf{P}) = \sigma \left(\sum_{i=1}^{N-n} \tilde{A}_{ji} \tilde{\lambda}_i^{-1} \sigma(\mathbf{B}_i v - \tilde{h}_i) + \mathbf{D}_j v - h_j \right) - \lambda_j v_j. \quad (6.6)$$

Note that the matrix \mathbf{C} is not involved in relation (6.6), which defines the family of the vector fields F (inertial forms). This property holds due to the property that inter-satellite interactions are dominated by the satellite-center ones. The next assertion means that this principle allows us to create a network dynamics with prescribed dynamics (if the network satisfies n/N_s -centrality assumption and N is large enough). It is valid under the additional condition that the interaction graph (V, E) verifies the centrality condition.

Theorem 6.2 *Assume n/N_s -centrality assumption is satisfied. Then the family of the vector fields F defined by (6.6) is dense in the set of all C^1 vector fields Q defined on the unit ball $B^n \subset \mathbb{R}^n$. In the other words, centralized Hopfield neural networks are maximally flexible.*

Let us choose some i_C such that i_C belongs to \mathcal{C} . The corresponding node will be called a controller hub. We introduce the control parameter ξ by

$$\lambda_{i_C} = \xi \bar{\lambda}_{i_C}, \quad (6.7)$$

where we fix a positive $\bar{\lambda}_{i_C}$.

Theorem 6.2 can be used to show the following

Theorem 6.3 (Maximal switchability theorem) *Let us consider dynamical systems (global semiflows) S_1^t, \dots, S_k^t , $t > 0$, defined on the n -dimensional closed ball $B^n \subset \mathbb{R}^n$ defined by finite dimensional vector fields $F^{(k)} \in C^1(B^n)$ and having structurally stable attractors \mathcal{A}_l , $l = 1, \dots, k$.*

For sufficiently large N and any graph (V, E) satisfying the n/N_s -centrality condition there exists a choice of interactions W_{ij} and thresholds h_i such that Assumption 1 holds and

- (i) *there exist a ξ_0 such that for all $\xi > \xi_0$ the dynamics of network (2.1) has a rest point, which is a global attractor;*
- (ii) *for an open interval of values ξ the global semiflow S_H^t defined by (2.1) have local attractors \mathcal{B}_l such that the restrictions of the semiflow S_H^t to \mathcal{B}_l are orbitally topological equivalent to the semiflows S_l^t restricted to \mathcal{A}_l .*

Finally, let us give an estimate on the maximal number of equilibria N_{eq} of centralized networks. This number is a characteristics of the network capacity, flexibility and adaptivity. To proceed to these estimates, let us define a procedure, which can be named decomposition into “distar” motifs. In the network interaction graph (E, V) we choose some nodes v_1, \dots, v_n , which we conditionally consider as hubs. By “distar” motif we understand a part of interaction graph consisting of the hub v_j and the subset S_j of the set S_j^* (defined by (2.5)) consisting of the nodes connected in both directions to v_j : $S_j = \{v_i \in V : (i, j) \text{ and } (j, i) \in E\}$. This distar motif becomes an usual star if directions of the edges are ignored. Consider the union U_n of all S_j . Some nodes $w \in U_n$ may belong to two different sets S_j and S_k , where $k \neq j$. We remove from the vertex set V all such nodes. After such removing we obtain a part of graph $G_n = (V', E')$ of the initial graph (E, V) , which is a union of n disjoint distars S_1, \dots, S_n , where each S_k contains a single center $\{v_k\}$ and $\mu(S_k)$ satellites connected with the center in both directions. Recall that the graph (V', E') is a part of graph (V, E) if $V' \subset V$ and $E' \subset E$. These numbers $\mu(S_k)$ depend on the choice of hub nodes $\{v_1, \dots, v_n\}$.

We will prove the following theorem:

Theorem 6.4 *The maximal possible number $N_{eq}(E, N)$ of equilibria of a network with a given interaction graph (E, V) , where V consists of N nodes, satisfies*

$$N_{eq} \geq \sup \mu(S_1)\mu(S_2)\dots\mu(S_n), \quad (6.8)$$

where the supremum is taken over all integers $n > 0$ and all graphs G_n , which are parts of interaction graph (V, E) and consist of n disjoint distars. Here $\mu(S_l)$ is the number of the nodes in the distar S_l .

Consider now graphs, which are unions of identical distars. The degree of the center of each distar is $\lfloor (N-n)/n \rfloor$. Then, the maximal possible number N_{eq} of equilibria in such a centralized network (2.1) with N nodes and n centers satisfies $N_{eq} \geq \lfloor (N-n)/n \rfloor^n$, where $\lfloor x \rfloor$ denotes the floor of a real number x . Note that for a fixed N the maximum of $(N/n)^n$ over $n = 1, 2, \dots$ is attained at $n = \lfloor N/5 \rfloor$, when the distars contain 5 satellites each. Therefore we obtain the estimate $N_{eq} \geq 4^{\lfloor N/5 \rfloor}$.

7 Proof of Theorem 6.1

Let us start by proving a lemma

Lemma 7.1 *Under assumptions (4.5), (4.6) and (4.7) for sufficiently small positive $\kappa < \kappa_0$ solutions (u, v) of (4.1), (4.2) and (4.3) satisfy*

$$w_i(t) = \kappa U_i(v(t), \mathbf{B}, \tilde{h}) + \tilde{w}_i(t), \quad (7.1)$$

where $U = (U_1, \dots, U_n)$ is defined by

$$U_i(v, \mathbf{B}, \tilde{h}) = \tilde{\lambda}_i^{-1} \sigma \left(\mathbf{B}_i v(t) - \tilde{h}_i \right). \quad (7.2)$$

Then, for some T_0 function \tilde{w} satisfies the estimates

$$|\tilde{w}(t)| < c_1 \kappa^2, \quad t > T_0 \quad (7.3)$$

where c_1 does not depend on t and κ . The time moment T_0 depends on initial data and the network parameters.

Proof. Let us introduce a new variables \tilde{w}_i by (7.1). They satisfy the equations

$$\frac{d\tilde{w}_i}{dt} = H_i(v, \tilde{w}) - \kappa^{-1} \tilde{\lambda}_i \tilde{w}_i, \quad (7.4)$$

where

$$H_i(v, \tilde{w}) = \kappa Z_i(v) + W_i(v, \tilde{w}),$$

$$Z_i(v) = \sum_{j=1}^n \frac{\partial U_i(v)}{\partial v_j} (\sigma(\bar{\mathbf{A}}_j U + \mathbf{D}_j v - h_j) - \xi \bar{\lambda}_j v_j),$$

and

$$W_i(v, \tilde{w}) = \sigma \left(\mathbf{B}_i v + \mathbf{C}_i \tilde{w} - \tilde{h}_i \right) - \sigma \left(\mathbf{B}_i v - \tilde{h}_i \right).$$

Let us estimate $H_i(v, \tilde{w})$ for sufficiently large t . According to (4.4), for such times we can use that $(w, v) \in \mathcal{B}(r(a), R(a))$, where $a > 1$. In this domain $\mathcal{B}(r(a), R(a))$ one has $\sup |Z_i| < c_2$ and $\sup |W_i| < c_3 \kappa$, where c_2, c_3 are independent of κ . Therefore,

$$H_i(v(t), \tilde{w}(t)) < c_0 \kappa, \quad t > T_0(\kappa, \mathbf{P}).$$

Now, as above in subsection 4.1, equation (7.4) entails estimate (7.3). The assertion is proved.

Proof of Theorem 6.1. The rest part of the proof of Theorem 6.1 uses the well known technique of invariant manifold theory, see, for example, [45, 38, 18]. Let us consider the domain $D_\kappa = \{w : |w| < c_1 \kappa^2\}$. Theorem 6.1.7 [18] shows that for $d \in (0, 1)$ there is a locally attractive C^{1+d} -smooth invariant manifold \mathcal{M}_n . Relation (6.3) follows from (7.3). The global attractivity of this manifold also follows from (7.3). The theorem is proved.

8 Proof of Theorems 6.2, 6.3 and 6.4

8.1 Proof of Theorem 6.2

The main idea of the subsequent statement is to study the dependence of the fields F_j defined by Eq.(6.6) on the parameters \mathbf{P} . To this end, we apply a special method stated in the next subsection.

Let us formulate a lemma, that gives us a key tool and which implies Theorem 6.2.

Lemma 8.1 *Assume*

$$a_i > \delta/\lambda_i, \quad b_i < (1 - \delta)/\lambda_i \quad i = 1, \dots, n. \quad (8.1)$$

Let $Q = (Q_1(v), \dots, Q_n(v))$ be a C^1 smooth vector field on $\Pi(a, b)$ and $\delta > 0$ verify

$$-\delta < Q_i(v) < \delta, \quad v \in \Pi(a, b), \quad i = 1, \dots, n. \quad (8.2)$$

Then there are parameters \mathbf{P} of the neural network such that the field F defined by (6.6) satisfies the estimates

$$\sup_{v \in \Pi(a, b)} |F(v, \mathbf{P}) - Q(v)| < \epsilon, \quad (8.3)$$

$$\sup_{v \in \Pi(a, b)} |\nabla F(v, \mathbf{P}) - \nabla Q(v)| < \epsilon. \quad (8.4)$$

In other words, the fields F are dense in the vector space of all C^1 smooth vector fields satisfying to (8.2).

Proof. The proof uses the standard results of the multilayered network theory.

Step 1. The first preliminary step is as follows. Let us solve the system of equations

$$\sigma(R_j) = Q_j(v) + \lambda_j v_j, \quad v \in \Pi(a, b) \quad (8.5)$$

with unknown R_j . Here R_j are the regulatory inputs of the sigmoidal functions. These equations have a unique solution due to conditions (2.2), (8.1) and (8.2): the right hand sides $Q_j + \lambda_j v_j$ range in $(0, 1)$. The solutions $R_i(v)$ are C^1 -smooth vector fields.

Step 2. Consider relation (6.6). We choose entries A_{ji} and B_{il} in a special way. First, let us set $A_{ji} = 0$ if $i \notin \mathcal{S}^*(j)$, where the set $\mathcal{S}^*(j)$ is defined in the n/N_s -centrality assumption, see condition ii. Recall that $\mathcal{S}^*(j)$ is the set of the satellites acting on the center j . Note that then sum (6.6) can be rewritten as

$$F_j(v, \mathbf{P}) = \sigma \left(\sum_{i \in \mathcal{S}^*(j)} \bar{A}_{ji} \tilde{\lambda}_i^{-1} \sigma(\mathbf{B}_i v - \tilde{h}_i) + \mathbf{D}_j v - h_j \right) - \lambda_j v_j. \quad (8.6)$$

Using the result of step 1 and this relation, we see that our problem is reduced to the following: to approximate $R_j(v)$ in C^1 norm with a small accuracy $O(\epsilon)$ by

$$H_j(v, \mathbf{P}) = \sum_{i \in \mathcal{S}^*(j)} \bar{A}_{ji} \tilde{\lambda}_i^{-1} \sigma(\mathbf{B}_i v - \tilde{h}_i) + \mathbf{D}_j v - h_j. \quad (8.7)$$

Note that, according to the centrality assumption, the set $\mathcal{S}^*(j)$ contains $N_s > CN^\theta$ elements. Moreover, due to this assumption, the sum $\mathbf{B}_i = \sum_k B_{ik} v_k$ involves all k , $k = 1, \dots, n$. Therefore, since n is fixed and N can be taken arbitrarily large, the theorem on the universal approximation by multilayered perceptrons (see, for example, [5]) implies that the fields $H = (H_1, \dots, H_n)$ are dense in the Banach space of all the vector fields on $\Pi(a, b)$ (with C^1 - norm). Therefore, H_j approximate R_j with $O(\epsilon)$ -accuracy in C^1 - norm. This finishes the proof.

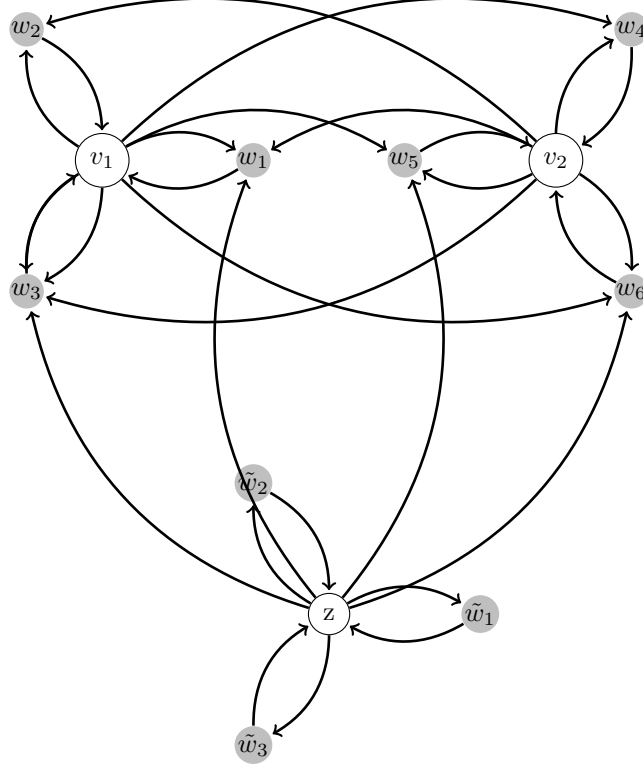


Figure 2: Modular architecture. The switching module consists of the center z and the satellites $\tilde{w}_1, \tilde{w}_2, \tilde{w}_3$. The generating module consists of the centers v_1, v_2 and the satellites w_1, \dots, w_6 .

8.2 Proof of Theorem 6.3

Ideas behind proof. Before stating a formal proof, we present a brief outline, which describes main ideas of the proof and the architecture of the switchable network. The network consists of two modules. The first module is a generating one and it is a centralized neural network with n centers v_1, \dots, v_n and satellites w_1, \dots, w_N . The second module consists of a center $v_{n+1} = z$ and m satellites $\tilde{w}_1, \dots, \tilde{w}_m$. The satellites from this module interact only with the module center z , i.e., in this module the interactions can be described by a distar graph. Only the center of the second module interacts with the neurons of the first (generating) module. We refer to the second module as a switching one. This architecture is shown on Fig. 2.

For the switching module the corresponding equations have the following form. Let us consider a distar interaction motif, where a node z is connected in both directions with m nodes $\tilde{w}_1, \dots, \tilde{w}_m$. We set $n = 1$ and $N_1 = m$, $\tilde{\lambda}_i = 1$, $\mathbf{D} = \mathbf{0}$, $\mathbf{C} = \mathbf{0}$, $\lambda_1 = 1$, and $A_{1j} = \kappa^{-1} \tilde{a}_j$ in eqs. (4.1) and (4.2). By such notation the equations for the switching module can be rewritten in the form

$$\frac{d\tilde{w}_i}{dt} = \sigma(\tilde{b}_i z - \tilde{h}_i) - \kappa^{-1} \tilde{w}_i, \quad (8.8)$$

$$\frac{dz}{dt} = \sigma\left(\kappa^{-1} \sum_{j=1}^m \tilde{a}_j \tilde{w}_j - h\right) - \xi \bar{\lambda} z, \quad (8.9)$$

where $i = 1, \dots, m$ and $\tilde{b}_i, \tilde{a}_j, \bar{\lambda} > 0$.

Under above assumptions on the network interactions, equations for generating module can be represented as follows:

$$\frac{dw_i}{dt} = \sigma(\mathbf{B}_i v + \mathbf{C}_i w - d_i z - \bar{h}_i) - \kappa^{-1} \tilde{\lambda}_i w_i, \quad (8.10)$$

$$\frac{dv_j}{dt} = \sigma(\mathbf{A}_j w + \mathbf{D}_j v - \tilde{d}_j z - h_j) - \lambda_j v_j, \quad (8.11)$$

where $i = 1, \dots, N$, $j = 1, \dots, m$ and d_i, \tilde{d}_j are coefficients.

These equations involve z as a parameter. This fact can be used in such a way. Consider the system of the differential equations

$$dv/dt = Q(v, z), \quad v = (v_1, \dots, v_n) \quad (8.12)$$

where z is a real control parameter. Let z_1, \dots, z_{m+1} be some values of this parameter. We find a vector field Q such that for $z = z_l$, where $l = 1, \dots, m$, the dynamics defined by (8.12) has the prescribed structurally stable invariant sets Γ_l . Furthermore, according to theorem 6.2, for each positive ϵ we can choose the parameters $N, \mathbf{B}_i, \mathbf{C}_i, \tilde{b}_i, \tilde{a}_i, \tilde{h}_i, \mathbf{A}_j, \mathbf{D}_j, d_i, \tilde{d}_j, h_j, \lambda_j, \tilde{\lambda}_i$ of the system (8.10) and (8.11) such that the dynamics of this system will have structurally stable invariant sets $\tilde{\Gamma}_l$ topologically equivalent to Γ_l .

For the switching module we adjust the center-satellite interactions and the center response time parameter ξ in such a way that for a set of values ξ the switching module has the dynamics of system (8.8), (8.9) with m different stable hyperbolic equilibria $z = z_1, z_2, \dots, z_{m+1}$ and for sufficiently large ξ system (8.8) and (8.9) has a single equilibrium close to $z_1 = 0$. Existence of such a choice will be shown in coming lemma 8.2. Then the both modules form a network having need dynamical properties formulated in the assertion of Theorem 6.3.

Proof. Let us formulate some auxiliary assertions. First we consider the switching module.

Lemma 8.2 *Let m be a positive integer and $\beta \in (0, 1)$. For sufficiently small $\kappa > 0$ there exist $\bar{a}_j, b_i, \tilde{h}_i, h$ such that*

- i** *for an open interval of values ξ system (8.8), (8.9) has m stable hyperbolic rest points $z_j \in (j - 1 + \beta, j + \beta)$, where $j = 1, \dots, m$;*
- ii** *for $\xi > \xi_0 > 0$ system (8.8), (8.9) has a single stable hyperbolic rest point.*

Proof. Let $h = 0$. To find equilibria z , we set $d\tilde{w}_i/dt = 0$, and express \tilde{w}_i via z . Then we obtain the following equation for the rest points z :

$$\xi z = \sigma \left(\sum_{j=1}^m \tilde{a}_j \sigma(\tilde{b}_j z - \tilde{h}_j) \right). \quad (8.13)$$

For especially adjusted parameters eq. (8.13) has at least m solutions, which give stable equilibria of system (8.8), (8.9). To show it, we assume that $0 < \kappa < 1$, $\tilde{b}_j = \tilde{b} = \kappa^{-1/2}$ and $\tilde{h}_j = \tilde{b}\mu_j$, where $\mu_j = j - 1 + \beta$. We obtain then

$$V(\xi z) = \sum_{j=1}^m \sigma(\tilde{b}(z - \mu_j)) + O(\kappa) = F_m(z, \beta, \kappa), \quad (8.14)$$

where $V(z)$ is a function inverse to $\sigma(z)$ defined on $(0, 1)$. Since $\tilde{b} \gg 1$ for small κ , the plot of the function F_m is close to a stairway (see Fig. 3). Let

$$\xi = 1, \quad \tilde{a}_1 = V(\mu_1) + \kappa, \quad \tilde{a}_j = V(\mu_j) - V(\mu_{j-1}), j = 2, \dots, m.$$

The intersections of the curve $V(z)$ with the almost horizontal pieces of the plot of F_m give us m stable equilibria of system (8.8),(8.9). These equilibria z_j lie in the corresponding intervals $(j - 1 + \beta, j + \beta)$. For sufficiently large ξ we have a single rest stable point z at 0. The lemma is proved.

Consider compact invariant hyperbolic sets $\Gamma_1, \dots, \Gamma_m$ of semiflows defined by arbitrarily chosen C^1 smooth vector fields $Q^{(l)}$ on the unit ball $B^n \subset \mathbb{R}^n$, where $l = 1, \dots, m$.

Lemma 8.3 *Let $\Pi(a, b)$ be a box in \mathbb{R}^n and $m > 1$ be a positive integer. There is a C^1 -smooth vector field Q on $\Pi(a, b) \times [0, m + 1]$ such that equation (5.1) defines a semiflow having hyperbolic sets $\Gamma_1, \dots, \Gamma_m$ and the restriction of this field on $\Pi(a, b) \times [0, 1]$ has an attractor consisting of a single hyperbolic rest point.*

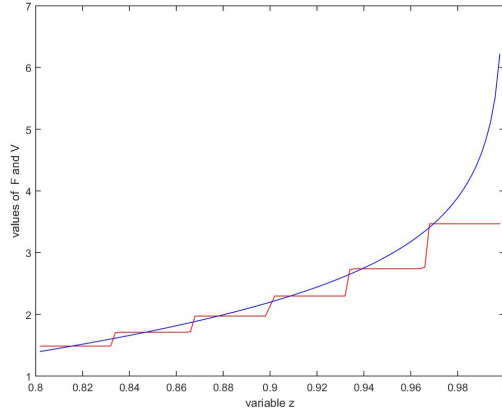


Figure 3: The intersections of the curve $F_m(z, \beta, \kappa)$ and the curve $V(z)$ give equilibria of system (8.8),(8.9) for $\xi = 1$. Stable equilibria correspond to the intersections of V with almost horizontal pieces of the graph of F_m .

Proof. The proof uses the following idea. For $k \in \{2, \dots, m + 1\}$ let $Q^{(k)}(v)$ be a vector field on $\Pi(a, b)$ having Γ_{k-1} as an invariant compact hyperbolic set. Moreover, suppose that $Q^{(1)}$ has a single globally attracting rest point in $\Pi(a, b)$, $z_j \in (j - 1 + \beta, j + \beta)$, where $j = 1, \dots, m$ and $\beta \in (0, 1)$. Let $\chi_k(z)$ be smooth functions of $z \in \mathbb{R}$ such that

$$\chi_k(z_l) = \delta_{lk}, \quad l \in \{1, \dots, m\}, \quad k = 1, \dots, m$$

where δ_{lk} stands for the Kronecker delta. Let $Q(v, z)$ be the vector field on $\Pi(a, b) \times [0, m + \beta]$ defined by

$$Q_i(v, z) = \sum_{k=1}^m Q_i^{(k)} \chi_k(z), \quad i \in \{1, \dots, n\}, \quad (8.15)$$

for first n components and $n + 1$ -th component of this field (denoted by z) is defined by

$$Q_{n+1}(v, z) = F_m(z, \beta, \kappa), \quad (8.16)$$

where F_m is defined by (8.14). For $\beta \in (0, 1)$ the function F_m has stable roots at the points $z = 1, 2, \dots, m$. We observe that the equation for z -component $dz/dt = F_m(z, \beta, \kappa)$ does not involve v . By applying Lemma 8.2 we note that solutions $z(t, z(0))$ of the Cauchy problem for this

differential equation verify $|z(t) - z_j| < \exp(-c_1 t)$, if $z(0)$ lies in an open neighbourhood of z_j . To conclude the proof, we consider the system

$$\begin{aligned} dv_i/dt &= Q_i(v, z), \quad i = 1, \dots, n, \\ dz/dt &= F_m(z, \beta, \kappa) - \xi \bar{\lambda} z = Q_{n+1}(z). \end{aligned}$$

The right hand sides of this system define the field Q of dimension $n + 1$ from the assertion of Lemma 8.3. To check this fact, we apply Lemmas 8.1 and 8.2 that completes the proof.

Next, to finish the proof of Theorem 6.3, let us take a box $\Pi(a, b)$, where $0 < a_i < b_i$. The semiflows defined by differential equations $dv/dt = \delta Q(v)$ are orbitally topologically equivalent for all $\delta > 0$. We approximate the first n components of the field Q by our neural network using Lemma 8.3. We multiply here Q on an appropriate positive δ to have a field with components bounded by sufficiently small number in order to apply Lemma 8.1. Namely, we take δ such that $a_i > \delta/(\xi_0 \bar{\lambda}_i)$ and $b_i < (1 - \delta)/(\xi_1 \bar{\lambda}_i)$ and apply Lemma 8.1. Note that this approximation does not involve the control parameter ξ . Indeed, this parameter is involved only in the approximation of Q_{n+1} , which can be done independently, see the distar graph lemma 8.2. This concludes the proof of Theorem 6.3.

Remark. In Theorem 6.3, we assume that the vector field $Q(v)$ is given. However, by centralized networks we can solve the problem of identification of dynamical systems supposing that the trajectories $v(t)$ are given on a sufficiently large time interval whereas Q is unknown or we know this field only up to unknown parameters. An example, where we consider an identification construction for a modified noisy Lorenz system, can be found in section 9.

8.3 Proof of Theorem 6.4

Let us refer to the distar centers as hubs and to periphery nodes as satellites. We suppose that satellites do not interact each with others and a satellite interacts only with the corresponding hub. Therefore the interaction graph resulting from the "hub disconnecting" construction consists of n disconnected distar motifs.

Step 1. Let $n = 1$. We apply lemma 8.2 to the distar graphs, see the proof of the previous theorem. Then we have m_1 stable equilibria, where m_1 is the number of satellites in the distar motif.

Step 2. In the case $n > 1$ we consider the disconnected interaction graph consisting of n distar motifs, where the j -th distar motif contains m_j nodes. One has $m_1 + m_2 + \dots + m_n = N - n$ and totally the graph consists of N nodes. For each distar we adjust the parameters as above (see step 1). We obtain thus $m_1 m_2 \dots m_n$ of equilibria and the theorem is proven.

9 Algorithm of construction of switchable network with prescribed dynamics

The proof of Theorem 6.3 can be used to construct practically feasible algorithms, which solve the problem of construction of a switchable network with prescribed dynamical properties. As a matter of fact, we can address two different, but related problems. The first problem is the *synthesis* of a neural network with prescribed attractors and switchability properties. The second problem is the *identification* of a neural network from time series. First we state the solution of the first problem and after we describe how to resolve the second one by analogous methods.

The prescribed network properties for the *synthesis problem* are stated in Theorem 6.3. We describe here a step by step algorithm, allowing to construct a network with these properties.

Consider structurally stable dynamical systems defined by the equations

$$dv/dt = Q^{(l)}(v) \quad v = (v_1, \dots, v_n) \in \Pi(a, b) \subset \mathbb{R}^n, \quad (9.1)$$

where $l = 1, \dots, m$ and $\Pi(a, b)$ is defined by (6.1). We suppose that the fields $Q^{(l)}(v)$ are sufficiently smooth, for example, $Q^{(l)} \in C^\infty(\Pi(a, b))$. Without any loss of generality we can assume that

$$1 < a_i < b_i, \quad (9.2)$$

(otherwise we can shift variables v_i setting $v_i = \tilde{v}_i - c_i$).

Step 1. Find a sufficiently small ϵ such that perturbations of vector fields $Q(v)^{(l)}$, which are ϵ small in C^1 norm, do not change topologies of semiflows defined by 9.1. Actually, it is hard to compute such a value of ϵ , so, in practice we simply choose a small ϵ by the trial and error method.

Step 2. We find a vector field $Q(v, z)$ with $n+1$ components, where $z = v_{n+1} \in [a_{n+1}, b_{n+1}] \subset \mathbb{R}$ such that the first n components of $Q(v, z)$ are defined by relations (8.15) and the $n+1$ component is defined by (8.16). Let $D = \Pi(a, b) \times [a_{n+1}, b_{n+1}]$.

To describe the next steps, first let us introduce the functions

$$G_j(\bar{v}, \mathbf{P}) = \sum_{i=1}^N \bar{A}_{ji} \sigma(\mathbf{B}_i \bar{v} - h_i), \quad (9.3)$$

where the parameter $\mathbf{P} = \{N, \bar{A}_{ji}, B_{ik}, h_j, j = 1, \dots, n+1, i, k = 1, \dots, N\}$ and $\bar{v} = (v_1, \dots, v_n, z)$.

Let us observe that dynamical systems $dq/dt = Q(q)$ and $dq/dt = \gamma Q(q)$ with $\gamma > 0$ have the same trajectories, invariant sets and attractors, therefore, instead of Q we can use γQ . We choose a $\gamma > 0$ and a small positive $\delta < 1$ such that

$$-\delta < \gamma Q_i(\bar{v}) < \delta, \quad \bar{v} \in D, \quad i = 1, \dots, n+1 \quad (9.4)$$

and

$$a_i > \delta/\lambda_i, \quad b_i < (1-\delta)/\lambda_i \quad i = 1, \dots, n+1 \quad (9.5)$$

for $\lambda_i > 1$.

Then (9.4) and (9.5) imply that

$$0 < \gamma Q_j(\bar{v}) + \lambda_j \bar{v}_j < 1, \quad \bar{v} \in D, \quad j = 1, \dots, n+1. \quad (9.6)$$

Let σ^{-1} be the function inverse to σ . Due to (9.6) the functions

$$R_j(\bar{v}) = \sigma^{-1}(\gamma Q_j(\bar{v}) + \lambda_j \bar{v}_j) \quad (9.7)$$

are correctly defined and smooth on D .

Now we solve the following approximation problem.

To find the number N , the matrices $\bar{\mathbf{A}}, \mathbf{B}$ and vector h such that

$$|R_j(\bar{v}) - G_j(\bar{v}, \mathbf{P})| + |D_{\bar{v}}(R_j(\bar{v}) - G_j(\bar{v}, \mathbf{P}))| \leq \epsilon/2, \quad j = 1, \dots, n+1. \quad (9.8)$$

This problem can be resolved by standard algorithms, which perform approximations of functions by multilayered perceptrons [5]. Note that these standard methods are based on iteration procedures, which can use a large running time.

We describe here a new variant of the algorithm for this approximation problem, which uses a wavelet-like approach. This approach does not exploit any iteration procedures or linear system solving. All the procedure reduces to a computation of the Fourier and wavelet coefficients. However, this algorithm is numerically effective only for sufficiently smooth R_j with fast decreasing Fourier coefficients and for not too large dimensions n .

The solution of the approximation problem (9.8) proceeds in the two steps.

Step 3. We reduce the $n + 1$ -dimensional problem (9.8) to a set of one-dimensional ones as follows. Let us approximate the functions R_j by the Fourier expansion:

$$\sup_{\bar{v} \in D} (|R_j(\bar{v}) - \hat{R}_j(\bar{v})| + |\nabla_{\bar{v}}(R_j(\bar{v}) - \hat{R}_j(\bar{v}))|) < \epsilon/4, \quad (9.9)$$

where

$$\hat{R}_j(\bar{v}) = \sum_{k \in K_D} \hat{R}_j(k) \exp(i(k, \bar{v})), \quad (9.10)$$

$(k, \bar{v}) = k_1 \bar{v}_1 + k_2 \bar{v}_2 + \dots + k_{n+1} \bar{v}_{n+1}$ and the set K_D of vectors k is a finite subset of the $(n + 1)$ -dimensional lattice L_D

$$K_D \subset L_D = \{k = (k_1, \dots, k_{n+1}) : k_i = (a_i - b_i)^{-1} \pi m_i \text{ for some } m_i \in \mathbb{Z}\}. \quad (9.11)$$

The Fourier coefficients $\hat{R}_j(k)$ can be computed by

$$\hat{R}_j(k) = (\text{volume}(D))^{-1} \int_D R_j(\bar{v}) \exp(-i(k, \bar{v})) d\bar{v}.$$

In order to satisfy (9.9), we take a sequence of extending sets K_D . For some K_D relation (9.9) will be satisfied because the Fourier coefficients $\hat{R}_j(k)$ fastly decrease in $|k|$.

Step 4. We exploit the fact that the problem (9.8) is linear with respect to the coefficients \bar{A}_{ij} . For each $k \in K_D$ we resolve the following one-dimensional problem. Let

$$g(q, M, a, \beta, \bar{h}) = \sum_{i=1}^M a_i \sigma(\beta_i(q - \bar{h}_i)). \quad (9.12)$$

We are seeking for integer $M > 0$ and the vectors $a = (a_1, \dots, a_M)$, $\beta = (\beta_1, \dots, \beta_M)$ and $\bar{h} = (\bar{h}_1, \dots, \bar{h}_M)$ such that

$$\sup_{q \in I_k} |W_{j,k}(q) - g(q, M, a, \beta, \bar{h})| < \epsilon(10|K_D|)^{-1}, \quad (9.13)$$

$$\sup_{q \in I_k} |dW_{j,k}(q)/dq - g'(q, M, a, \beta, \bar{h})| < \epsilon_1 \leq \epsilon(10|K_D|)^{-1}, \quad (9.14)$$

where $|K_D|$ is the number of the elements k in the set K_D ,

$$W_{j,k}(q) = \hat{R}_j(k) \exp(iq),$$

$$g'(q, M, a, \beta, \bar{h}) = \sum_{i=1}^M a_i \sigma'(\beta_i(q - \bar{h}_i)), \quad (9.15)$$

and $q = (k, \bar{v}) \in I_k$, where I_k is the interval $[q_-(k), q_+(k)]$ with

$$q_-(k) = \min_{\bar{v} \in D} (k, \bar{v}), \quad q_+(k) = \max_{\bar{v} \in D} (k, \bar{v}).$$

These approximation problems are indexed by (j, k) , where $j = 1, \dots, n + 1$ and $k \in K_D$ (we temporarily omit dependence on (j, k) in a, β, \bar{h}, M to simplify notation).

To resolve these one-dimensional approximation problems, we apply a method based on the wavelet theory. Notice that this method is numerically effective. First we observe that if (9.14) is fulfilled with a sufficiently small ϵ_1 , then, to satisfy (9.13), it is sufficient to add a constant term of the form $a_{M+1} \sigma(b_{M+1} q)$ with $b_{M+1} = 0$ to the sum in the right hand side of (9.12).

Let us define the function ψ by

$$\psi(q) = \sigma'(q) - \sigma'(q-1). \quad (9.16)$$

We observe that

$$\int_{-\infty}^{\infty} \psi(q) dq = 0 \quad (9.17)$$

and $\psi(q) \rightarrow 0$ as $|q| \rightarrow \infty$, therefore, ψ is a wavelet-like function.

Let us introduce the following family of functions indexed by the real parameters r, h :

$$\psi_{r,\xi}(q) = |r|^{-1/2} \psi(r^{-1}(q - \xi)). \quad (9.18)$$

For any $f \in L_2(\mathbb{R})$ we define the wavelet coefficients $T_f(r, \xi)$ of the function f by

$$T_f(r, \xi) = \langle f, \psi_{r,\xi} \rangle = \int_{-\infty}^{\infty} dq f(q) \psi_{r,\xi}(q). \quad (9.19)$$

For any smooth function f with a finite support $I_R = (-R, R)$ one has the following fundamental relation:

$$f = c_\psi \int_0^\infty \int_{-\infty}^\infty r^{-2} dr d\xi T_f(r, \xi) \psi_{r,\xi} = f_{wav}. \quad (9.20)$$

for some constant c_ψ . This equality holds in a weak sense: the left hand side and the right hand side define the same linear functionals on $L_2(\mathbb{R})$, i.e., for each smooth, well localized g one has

$$\langle f, g \rangle = \langle f_{wav}, g \rangle.$$

Let $\delta(\epsilon) \ll \epsilon$ be a small positive number. According to (9.20) we can find positive integers p_1, p_2 , points $r_1, \dots, r_{p_1}, \xi_1, \dots, \xi_{p_2}$ and a constant \bar{c}_ψ such that the integral in the right hand side of (9.20) can be approximated by a finite sum:

$$\sup |f(q) - \bar{f}_{wav}(q)| < \delta, \quad (9.21)$$

where

$$\bar{f}_{wav} = \bar{c}_\psi \sum_{l_1=1}^{p_1} \sum_{l_2=1}^{p_2} r_{l_1}^{-2} T_f(r_{l_1}, \xi_{l_2}) \psi_{r_{l_1}, \xi_{l_2}}.$$

In our case for each (j, k) we set $f = W_{j,k}(q)$ for $q \in I_k$ and $f = 0$ for $q \notin I_k$. We can take $r_{l_1} = r_+ l_1 / p_1$, where r_+ is large enough, and $\xi_{l_2} = q_{\min} + (q_{\max} - q_{\min}) l_2 / p_2$, where $q_{\min} < q_-(k)$, $q_{\max} > q_+(k)$ are sufficiently large and $l_1 = 1, \dots, p_1, l_2 = 1, \dots, p_2$. We can renumerate the points (r_{l_1}, ξ_{l_2}) by a single index $l = 1, \dots, p$, where $p = p_1 p_2$, that gives us r_l, ξ_l and the wavelet coefficients $T_l = \bar{c}_\psi T_f(r_l, \xi_l)$.

Having p, r_l, ξ_l and the wavelet coefficients T_l , we obtain the following solution of the approximation problem (9.12):

$$M(j, k) = p, \quad \bar{h}_{2l-1}(j, k) = r_l^{-1} \xi_l, \quad \bar{h}_{2l}(j, k) = r_l^{-1} (\xi_l + 1),$$

$$\beta_{2l-1}(j, k) = \beta_{2l}(j, k) = r_l^{-1}, \quad a_{2l-1}(j, k) = -a_{2l}(j, k) = T_l,$$

where we have introduced the index (j, k) in notation for the solution (M, a, β, \bar{h}) to emphasize that problem (9.12) depends on this index.

Finally, in the end of this step we obtain the coefficients

$$M(j, k), a_1(j, k), \dots, a_{M(j,k)}(j, k), \beta_1(j, k), \dots, \beta_{M(j,k)}(j, k), \bar{h}_1(j, k), \dots, \bar{h}_{M(j,k)}(j, k). \quad (9.22)$$

Step 5. We construct a network with $n + 1$ centers $\bar{v}_1, \dots, \bar{v}_{n+1}$ and N satellites as follows. Let $\mathbf{C} = 0$ and $\mathbf{D} = 0$, i.e., we assume that the satellites don't interact among themselves and there are no direct interactions between the centers. The number of satellites is defined by

$$N = \sum_{j=1}^{n+1} \sum_{k \in K_D} M(j, k).$$

Each satellite can be equipped with a triple index (i, j, k) , where $j = 1, \dots, n + 1$, $k \in K_D$ and $i \in \{1, \dots, M(j, k)\}$. We set that all $h_j = 0$, $\tilde{\lambda}_i = 1$, and λ_j are chosen as above. The threshold $h_{i,j,k}$ for the satellite with the index (i, j, k) is defined by

$$h_{i,j,k} = \bar{h}_i(j, k)$$

where $\bar{h}_i(j, k)$ are obtained at the Step 4 (see (9.22)).

Furthermore, we define the matrices \mathbf{A} and \mathbf{B} as follows. One has

$$B_{(i,j,k),l} = \beta_i(j, k)k_l,$$

(this relation describes an action of the l -th center on the satellite with index (i, j, k)) and

$$\bar{A}_{l,(i,j,k)} = a_l(j, k)$$

(this relation describes an action of the l -th center on the satellite with index (i, j, k)). Here $i \in \{1, \dots, M(j, k)\}$, $j, l = 1, \dots, n + 1$ and $k \in K_D$.

Remark. This algorithm can be simplified if instead networks (4.1), (4.2) we use analogous networks where satellites act on centers in a linear way:

$$\frac{dw_i}{dt} = \sigma \left(\mathbf{B}_i v + \mathbf{C}_i w - \tilde{h}_i \right) - \kappa^{-1} \tilde{\lambda}_i w_i, \quad (9.23)$$

$$\frac{dv_j}{dt} = (\mathbf{A}_j w - h_j) - \lambda_j v_j, \quad (9.24)$$

where $i = 1, \dots, N_1$, $j = 1, \dots, n$, and the fields $Q^{(l)}$ are defined by polynomials (note that Jackson's theorems [1] guarantee that any Q can be approximated by a polynomial field on $\Pi(a, b)$ in C^1 -norm). Then we can simplify Step 3 and Step 4 of the algorithm as follows. We observe that we can set $\gamma = 1$ and in this case the functions R_j have the form

$$R_j(\bar{v}) = Q_j(\bar{v}) + \lambda_j \bar{v}_j. \quad (9.25)$$

On Step 3 for polynomial functions $R_j(v)$ we can also use simple algebraic transformations, instead of the Fourier decomposition, to reduce the multidimensional approximation problem to one dimensional ones. On step 4 the function ψ defined by (9.16) is well localized and therefore alternatively step 4 can be realized by standard programs using radial basic functions and the method of least squares (see an example on the Lorenz system below).

Let us turn now to the problem of *identification* of a neural network from time series produced by a dynamical system $dv/dt = Q(v, \mathbf{P})$, $v \in \mathbb{R}^n$ with unknown parameters \mathbf{P} . Assume that we observe a time series $v(t_1), v(t_2), \dots, v(t_K)$ and the time interval between observations is small: $t_{i+1} - t_i = \Delta t \ll 1$. We want to construct a network with n centers, which produces, in a sense, analogous time series. According to (3.4), a suitable criterion of trajectory similarity is as follows. We can approximate the averages $S_{Q,\phi}$ from (3.3) by the time series

$$S_{Q,\mathbf{P},\phi} \approx K^{-1} \Delta T \sum_{k=1}^K \phi(v(t_k)) = S_{Q,\mathbf{P},\phi}^{(K)}. \quad (9.26)$$

Then, if the network identification is correct, the averages defined by time series and the corresponding ones generated by the approximating centralized neural network, should be close for smooth weight functions ϕ :

$$|S_{Q, \mathbf{P}, \phi}^{(K)} - S_{G_{anN}, \phi}^{(K)}| = Err_{approx} < \delta(\phi) \ll 1, \quad (9.27)$$

where G_{anN} is the approximation of Q by the neural network.

As a first step, we can approximate the unknown field $Q(v)$ by finite differences, for example, using the relation

$$Q(\tilde{v}_i, \mathbf{P}) = (v(t_{i+1}) - v(t_i))\Delta t^{-1}, \quad \tilde{v}_i = (v(t_{i+1}) + v(t_i))/2. \quad (9.28)$$

For other values v the field Q can be reconstructed, for example, by a linear interpolation. The neural network approximation of Q can be obtained by applying the steps 2-5 of the synthesis algorithm described above.

We end this section with an illustration of the simplified variant of the identification and synthesis algorithm, see the preceding Remark.

As an example, we describe a solution of the following identification problem. Consider time series generated by the Lorenz system perturbed by noise. The Lorenz system involves a *controller* parameter. Adjusting the values of this parameter, we can obtain chaotic dynamics, time periodic one or dynamics with convergent trajectories. We are going to find a centralized network, which also has a controller parameter and can generate all this rich variety of trajectories. For chaotic and periodic trajectories this neural approximation should exhibit dynamics with analogous ergodic properties (in the sense of (9.27)).

Recall that the Lorenz system has the form

$$dx/dt = \alpha(y - x), \quad dy/dt = x(\rho - z) - y, \quad dz/dt = xy - \beta z. \quad (9.29)$$

This system shows a chaotic behaviour for $\alpha = 10, \beta = 8/3$ and $\rho = 28$. For $\alpha = 10, \beta = 8/3$ and $\rho \in (0, 1)$ this system has a globally attracting rest point.

We introduce new variables $v_1 = x, v_2 = y, v_3 = z$ and $v_4 = \rho$ and consider a more complicated modified Lorenz system with a controller parameter: (compare with the proof of Theorem 6.3):

$$dv_1/dt = \alpha(v_2 - v_1) = f_1, \quad dv_2/dt = r_1 v_1(v_4 - v_3) - r_2 v_2 = f_2, \quad (9.30)$$

$$dv_3/dt = r_3 v_1 v_2 - \beta z = f_3, \quad dv_4/dt = \sigma_H(v_4, b_0, h_0) - \xi v_4 = f_4, \quad (9.31)$$

where σ_H is a regularized step function defined by $H_1(w) = (1 + \exp(-b_0(w - h_0)))^{-1}$ with $b_0 \gg 1$ and $h_0 = 1$. We set $\xi = 0.5, r_1 = 14, r_2 = 1, r_3 = 1$. The initial data for the fourth component $v_0 = v_4(0)$ is a controller parameter. For large b_0 the differential equation for v_4 has two stable equilibria: $v_4^- \approx 0$ and $v_4^+ \approx 2$. Therefore, for $v_0 \in (0, 1)$ system (9.30), (9.31) has a globally attracting rest point and for $v_0 > 1$ the attractor of this system is chaotic Lorenz one. The parameters of this system are $\mathbf{P} = (\alpha, \beta, r_1, r_2, r_3)$.

Suppose we observe trajectories $v(t), t \in [0, T]$ of system (9.30) at some time moments $t_0 = 0, t_1 = dt, \dots, t_p = p\Delta t$. In order to simulate experimental errors we have perturbed the system with additive noise. We are going to find a centralized network, which has an attractor with, in a sense, similar statistical characteristics. More precisely, we aim to minimize Err_{approx} from relation (9.27). For identification procedure we use a centralized network with 4 centers v_1, v_2, v_3 and v_4 . In this case steps 3, 4 can be simplified if we use this specific form of the modified Lorenz system. The last center v_4 serves as a controller.

We state the algorithm for the modified Lorenz system, however, the method is general and feasible for identification by trajectories generated by all low-dimensional dynamical systems defined by polynomial vector fields.

First we set

$$\mathbf{C} = \mathbf{D} = 0. \quad (9.32)$$

This means that only satellites act on centers and vice versa. To find the matrices \mathbf{A} , \mathbf{B} and the thresholds h_i , we solve the following approximation problems:

$$R(\mathbf{A}, \mathbf{B}, h) \rightarrow \min, \quad R = \sum_{i=1}^4 \sum_{j=1}^p (Q_i(t_j) - S_i(v(t_j), \mathbf{A}, \mathbf{B}, h))^2 \quad (9.33)$$

where

$$Q_i(t_j) = (v_i(t_j + \Delta t) - v_i(t_j))/\Delta t, \quad S_i(v, \mathbf{A}, \mathbf{B}, h) = \sum_{k=1}^{N_i} A_{ik} \sigma \left(\sum_{j=1}^p B_{kj} v_j - h_{ik} \right). \quad (9.34)$$

This approximation problem is nonlinear with respect to B and h . We can simplify this problem by the following heuristic method. Each function $f_i(\mathbf{v})$ defined on a open bounded domain can be represented as a linear combination of functions $g_l(\mathbf{v} \cdot \mathbf{k}_{li})$, where vectors \mathbf{k}_{li} belong to a finite set of vectors K_i . For example, for system (9.30), (9.31) the components f_j for $j = 1, 2, 3$ can be represented as linear combinations of monomials:

$$f_j(v) = g_j(v) - \lambda_j v_j, \quad g_j(v) = \sum_{l=1}^{11} C(j, l) T_l(v) \quad (9.35)$$

where

$$T_l = v_l, \quad l = 1, 2, 3, 4 \\ T_{2l+1} = (v_1 + v_l)^2, \quad T_{2l+2} = (v_1 - v_l)^2, \quad l = 2, 3, 4, \quad T_{11} = 1.$$

and $\lambda_1 = \alpha$, $\lambda_2 = 1$, $\lambda_3 = \beta$. Therefore, $K_1 = \{\mathbf{k}_{11} = (1, 0, 0, 0)\}$, $K_2 = \{\mathbf{k}_{12} = (1, 0, 1, 0), \mathbf{k}_{22} = (1, 0, -1, 0), \mathbf{k}_{32} = (1, 0, 0, 1), \mathbf{k}_{42} = (1, 0, 0, -1)\}$, $K_3 = \{\mathbf{k}_{13} = (1, 1, 0, 0), \mathbf{k}_{23} = (1, -1, 0, 0)\}$, $K_4 = \{\mathbf{k}_{14} = (1, 0, 0, 0)\}$. Let n_i be the number of the vectors contained in the set K_i , $n_1 = 1, n_2 = 4, n_3 = 2$ and $n_4 = 1$. In this case of the modified Lorenz system, the set K_D from (9.11) is the union of sets K_i , $i = 1, \dots, 4$.

We take a sufficiently large N_L , a large b_0 and define the auxiliary thresholds $\bar{h}_{\mathbf{k}_{li}, j}$, where $j = 1, \dots, N_L$, by

$$\bar{h}_{\mathbf{k}_{li}, j} = \min_{s=1, \dots, p, l \in K_i} v(t_s) \cdot \mathbf{k}_{li} + j \left(\max_{s=1, \dots, p, l \in K_i} v(t_s) \cdot \mathbf{k}_{li} - \min_{s=1, \dots, p, l \in K_i} v(t_s) \cdot \mathbf{k}_{li} \right) / N_L.$$

We seek coefficients $\bar{A}_{il, \mathbf{k}_{li}}$ and C_i , which minimize $R_i(\bar{\mathbf{A}}, C_i)$ for $i = 1, 2, 3, 4$:

$$R_i(\bar{\mathbf{A}}, C_i) \rightarrow \min, \quad R_i = \sum_{j=1}^p (Q_i(t_j) - \tilde{S}_i(v(t_j), \bar{\mathbf{A}}, C_i))^2 \quad (9.36)$$

where

$$\tilde{S}_i(v, \bar{\mathbf{A}}, C) = C_i + \sum_{l=1}^{n_i} \sum_{j=1}^{N_L} \bar{A}_{ij, \mathbf{k}_{li}} \sigma(b_0(\mathbf{k}_{li} \cdot v - \bar{h}_{\mathbf{k}_{li}, j})). \quad (9.37)$$

Note that since \tilde{S}_i are linear functions of $\bar{A}_{il, \mathbf{k}_{li}}$ and C_i , problems (9.36) can be solved by the least square method. The important advantage of this approach is that approximations can be done independently for different components i .

This approximation produces a centralized network involving 4 centers and $N = 8N_L + 8$ satellites. Indeed, each vector \mathbf{k}_{li} associated with a quadratic term T_l , gives us N_L satellites to

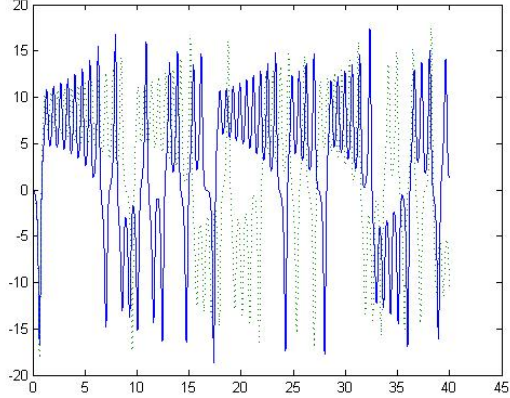


Figure 4: This plot shows trajectories of v_1 -component of the Lorenz system perturbed by noise (the solid curve) and its neural approximation with $N = 20$ satellites (the dotted curve). The curves are not close but they exhibit almost identical statistical properties ($Err_{approx} = 0.008$ (the white noise level is 0.05, solutions have been obtained by the Euler method with the time step 0.001 on the interval $[0, 40]$)).

approximate this term. Moreover, we use 4 satellites for approximations of the linear terms and 4 satellites are necessary for constants C_i in the right hand sides of (9.37).

The numerical simulations give the following results. The trajectories to identify are produced by the Euler method applied to the system (9.30), (9.31) perturbed by noise, where the time step 0.005 on the interval $[0, 50]$, the noise is simulated by $\epsilon_N \omega(t_i)$, where $\omega(t)$ is the standard white noise and $\epsilon_N = 0.05$. As a result of minimization procedure, we have obtained the errors R_i of the order 0.01 – 0.1. The trajectories of the system (9.30), (9.31) perturbed by noise and the corresponding neural networks are not close but they have a similar form and statistical characteristics that is confirmed by the value Err_{approx} (defined by (9.27)), which is 0.008, where the test function ϕ is $\phi(v) = v_1^2 + v_2^2/2 - 2v_3$. These results are illustrated by Fig. 4.

10 Conclusion and discussion

In this paper, we have proposed a complete analytic theory of maximally flexible and switchable Hopfield networks. We shown that dynamics of a network with n slow components v_1, \dots, v_n can be reduced to a system of n differential equations defined by a smooth n dimensional vector field $F(v)$. If these slow components are hubs, i.e., they are connected with a number of other weakly connected nodes (satellites) and center-satellite interactions dominate inter-satellite forces, then the network becomes maximally flexible. Namely, by adjusting only center-satellite interactions we can obtain smooth F of arbitrary forms.

These networks are also maximally switchable. We describe networks of a special architecture, which contains a controller hub. By changing the state of this hub and the hub response time parameter ξ one can completely change the network dynamics from an unique global attractive steady state to any combination of periodic or chaotic attractors.

Our results provide a rigorous framework for the idea that centralized networks are flexible. We also propose mechanisms for switching between attractors of these networks with controller hubs. In functional genomics there are numerous examples when transitions between attractors of gene regulatory networks can be triggered by controller proteins having multiple states sometimes resulting from interactions with micro-RNA satellites [8]. Similarly, neurons having multiple inter-

nal states can trigger phase transitions of brain networks suggesting that single neuron activation could be used for neural network control [15].

The proofs of our results are constructive and are based on an algorithm allowing the network reconstruction. This algorithm has several potential applications in biology. Identified networks can be used to study emergent network properties such as robustness, controllability and switchability. Gene networks with the desired switchability properties could be build by synthetic biology tools for various applications in biotechnology. Furthermore, maximal switchable network models can be used in neuroscience to relate structure and function in the brain activity, or in genetics to explain how a minimal number of mutations can induce large phenotypic changes from one type of adaptive behavior to another one.

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